Lecture 11 – Probability

DSC 10, Fall 2025

Dis today
Quiz 2 Weds

Agenda

We'll cover the basics of probability theory. This is a math lesson; take written notes ✍🏽.

Probability resources

Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5**.
- **Theory Meets Data, Chapters 1 and 2**.
- **Khan Academy's unit on Probability**.

Probability theory

- Some things in life *seem* random.
    - e.g., flipping a coin or rolling a die 🎲.
- The **probability** of seeing "heads" when flipping a fair coin is $\frac{1}{2}$.
- One interpretation of probability says that if we flipped a coin infinitely many times, then $\frac{1}{2}$ of the outcomes would be heads.

Terminology

- **Experiment**: A process or action whose result is random.
  - e.g., rolling a die.
  - e.g., flipping a coin twice.
- **Outcome**: The result of an experiment.
  - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
  - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event**: A set of outcomes.
  - e.g., the event that the die lands on a even number is the set of outcomes {2, 4, 6}.
  - e.g., the event that the die lands on a 5 is the set of outcomes {5}.
  - e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.

Terminology

- **Probability**: A number between 0 and 1 (equivalently, between 0% and 100%) that describes the likelihood of an event.
    - 0: The event never happens.
    - 1: The event always happens.
- Notation: If A is an event, $P(A)$ is the probability of that event.

$P(\text{even \# when roll die}) = \frac{3}{6}$

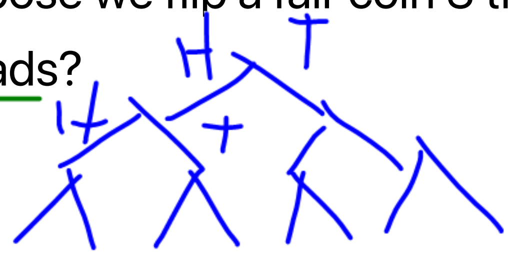
1	2
3	4
5	6

Equally-likely outcomes

- If all of the possible outcomes are equally likely, then the probability of A is

$$P(A) = \frac{\#\text{ of outcomes satisfying } A}{\text{total } \#\text{ of outcomes}}$$

- **Example 1**: Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?



$8 = 2 \times 2 \times 2$

outcomes

TTT
TTH
THT
THH
HTT
HTH
HHT
HHH

$\frac{3}{8}$

~~not ok~~

~~outcomes~~

~~0H 1H~~
~~2H 3H~~

~~$\frac{1}{4}$~~

multiply

Concept Check ✅ – Answer at **cc.dsc10.com**

A

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then – **without putting it back** – I choose another card at random and it is red?

B

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

$$\underbrace{\frac{1}{3}}_{\text{green}} * \underbrace{\frac{1}{2}}_{\text{red}} = \frac{1}{6}$$

$P(A)\ P(B \text{ given } A)$

outcomes

GR	GB
RB	RG
BR	BG

$\frac{1}{6}$ direct

Conditional probabilities

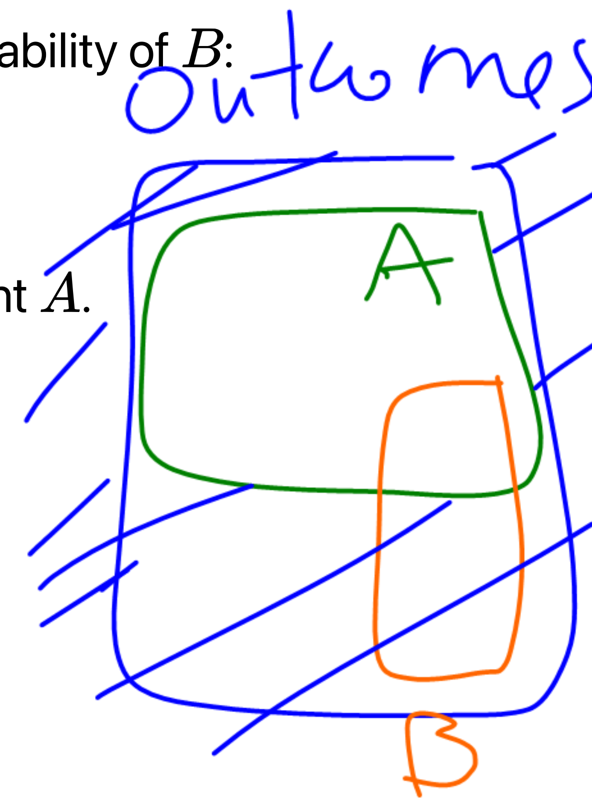
- Two events A and B can both happen. Suppose that we know A has happened, but we don't know if B has.
- If all outcomes are equally likely, then the conditional probability of B given A is:

$$P(B \text{ given } A) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\#\text{ of outcomes satisfying } A}$$

- Intuitively, this is similar to the definition of the regular probability of B:

$$P(B) = \frac{\#\text{ of outcomes satisfying } B}{\text{total }\#\text{ of outcomes}}$$

if you restrict the set of possible outcomes to just those in event A.



A = know it happened

B = wondering if it happened

Concept Check ✅ – Answer at **cc.dsc10.com**

$$P(B \text{ given } A) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\#\text{ of outcomes satisfying } A}$$

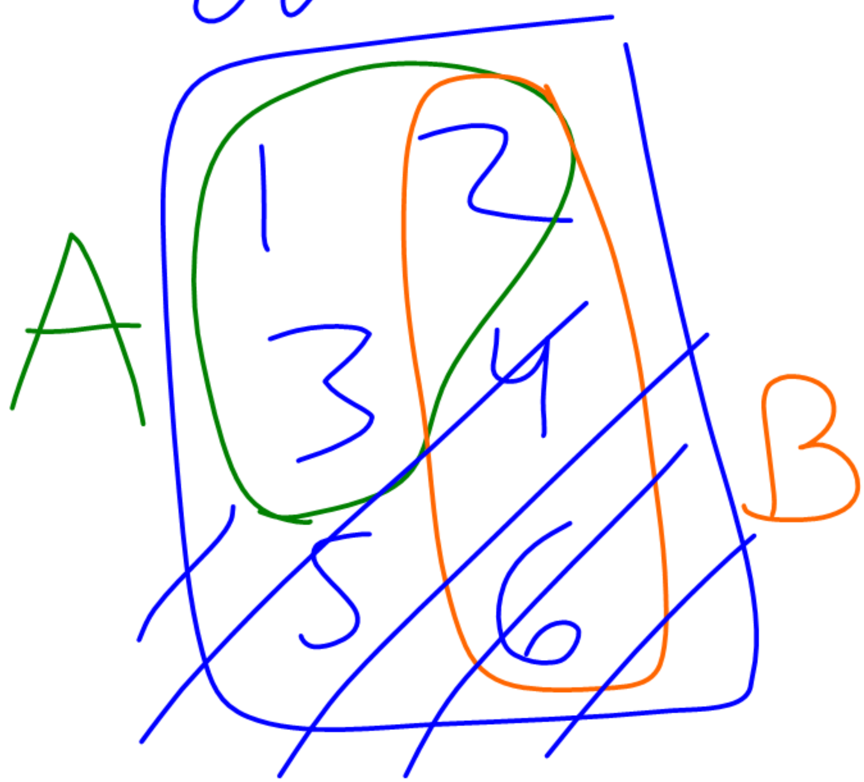
A

I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

- A) $\frac{1}{2}$
- B) $\frac{1}{3}$
- C) $\frac{1}{4}$
- D) None of the above.

B

outcomes



$\frac{1}{3}$

Probability that two events both happen

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that both A and B occur is

$$P(A \text{ and } B) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\text{total } \#\text{ of outcomes}}$$

- **Example 2**: I roll a fair six-sided die. What is the probability that the roll is 3 or less **and** even?

$\frac{1}{6}$

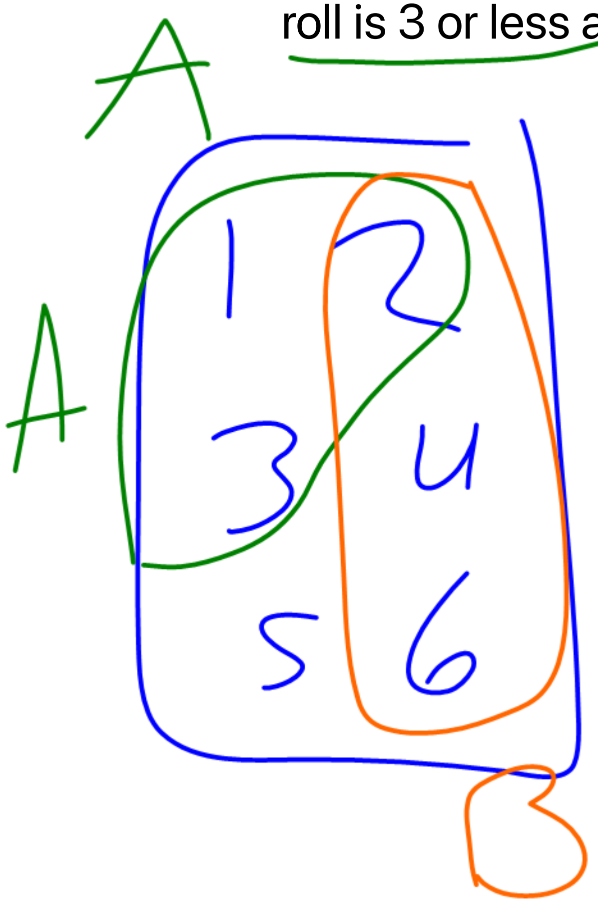
direct

The multiplication rule

- The multiplication rule specifies how to compute the probability of both A and B happening, even if all outcomes are not equally likely.

$$\underbrace{P(A \text{ and } B)} = \underbrace{P(A)} \cdot \underbrace{P(B \text{ given } A)}$$

- **Example 2, again**: I roll a fair six-sided die. What is the probability that the roll is $\underbrace{\text{3 or less}}_{A}$ and $\underbrace{\text{even}}_{B}$?



$$\frac{\cancel{3}}{6} * \frac{1}{\cancel{3}} = \frac{1}{6}$$

What if A isn't affected by B? 🤔

- The multiplication rule states that, for any two events A and B,

$$P(A \text{ and } B) = P(A) \cdot P(B \cancel{\text{ given } A})$$

when independent

- What if knowing that A happens doesn't tell you anything about the likelihood of B happening?
    - Suppose we flip a fair coin three times.
    - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?

flip 2 coins

$P(\text{H on 1st and T on 2nd})$

$\frac{1}{4}$

HH	HT
TH	TT

Independent events definition

independence

- Two events A and B are independent if $P(B \text{ given } A) = P(B)$ or equivalently if

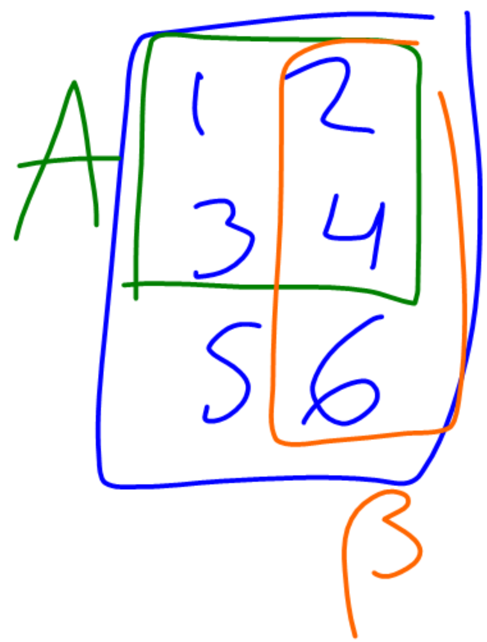
A = roll ≤ 4

B = even

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

- **Example 3**: Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

$P(H_{\text{on 1st flip}} \text{ AND } H_{\text{on 2nd flip}} \text{ AND } \ldots)$



$0.7 \times 0.7 \times \ldots$

$\boxed{0.7^5}$

Probability that an event *doesn't* happen

- The probability that A **doesn't** happen is $1 - P(A)$.
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.

Concept Check ✅ – Answer at **cc.dsc10.com**

Y = yes N = no

Every time I call my grandma 👵, the probability that she answers her phone is $\frac{1}{3}$, independently for each call. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- ~~A) $\frac{1}{3}$~~
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- ~~D) 1~~
- E) None of the above.

8 outcomes = 2 × 2 × 2

YYY

YYN $\rightarrow \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3}$

YNY

YNN

NYY

NYN

NNY

NNN $\rightarrow \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$

$$1 - \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$$

$$1 - \frac{8}{27}$$

$$\frac{19}{27}$$

not ok: $\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$

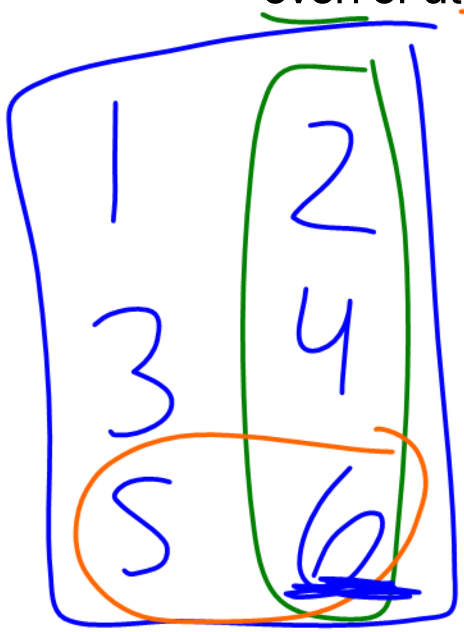
even ≥5

Probability of either of two events happening

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that either A or B occur is

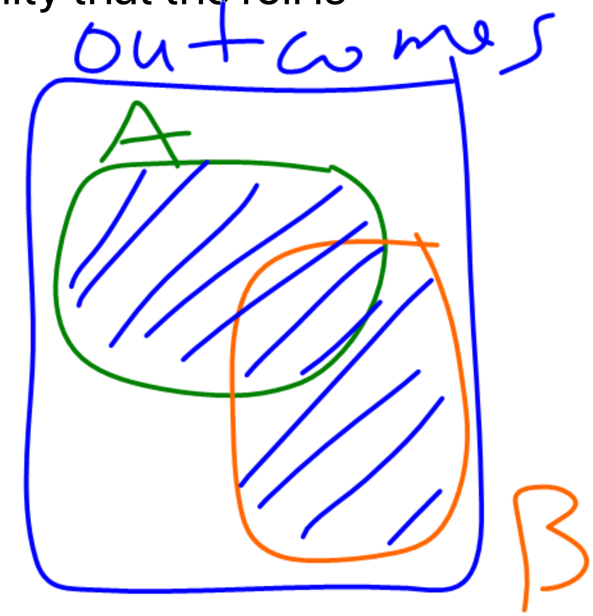
$$P(A \text{ or } B) = \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total }\#\text{ of outcomes}}$$

- **Example 4**: I roll a fair six-sided die. What is the probability that the roll is even or at least 5?



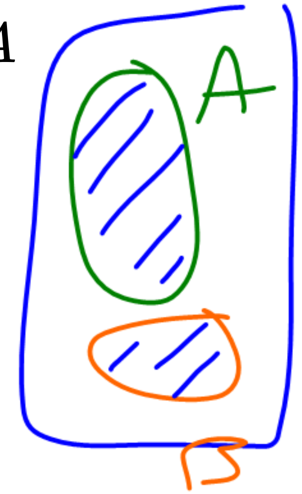
$\frac{4}{6}$

in general can't replace OR with +



The addition rule

- Suppose that if A happens, then B doesn't, and if B happens, then A doesn't.
    - Such events are called **mutually exclusive** – they have **no overlap**.
- If A and B are any two mutually exclusive events, then



$$P(A \text{ or } B) = P(A) + P(B)$$

- **Example 5**: Suppose I have two biased coins, a red coin and a blue coin. The red coin flips heads with probability 0.6, and the blue coin flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

$$P\left(\underbrace{\text{red } H \text{ and blue } T}_{0.6 \times 0.7} \text{ OR } \underbrace{\text{red } T \text{ and blue } H}_{0.4 \times 0.3}\right)$$

$$0.6 \times 0.7 + 0.4 \times 0.3$$

$$0.54$$

Aside: Proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$
\begin{aligned}
P(A \text{ or } B) &= \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total }\#\text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A) + (\#\text{ of outcomes satisfying } B)}{\text{total }\#\text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A)}{\text{total }\#\text{ of outcomes}} + \frac{(\#\text{ of outcomes satisfying } B)}{\text{total }\#\text{ of outcomes}} \\
&= P(A) + P(B)
\end{aligned}
$$

Summary, next time

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
    - The **multiplication rule**, which states that for any two events, $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$.
    - The **addition rule**, which states that for any two **mutually exclusive** events, $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** Simulations.