Lecture 11 – Probability

DSC 10, Spring 2024

Announcements

- Discussion is this afternoon. Problems are **here**.
- Lab 3 is due **tomorrow at 11:59PM**.
- Quiz 2 is on **Friday** in your assigned quiz session.
    - You should get an email tomorrow with your seating assignment.
    - Bring your ID and a pencil.
    - This is a 20 minute paper-based quiz with no aids allowed.
    - The quiz covers Lecture 5 through 9 and related labs and homeworks.
    - Quiz 2 is **more challenging** than Quiz 1, and next week's Midterm Exam will be more challenging than Quiz 2. 📈
- Homework 3 is due on **Tuesday**.
    - Problems 1 and 2 only are relevant to Quiz 2.

Agenda

We'll cover the basics of probability theory. This is a math lesson; take written notes ✍🏽.

Probability resources

Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5**.
- **Theory Meets Data, Chapters 1 and 2**.
- **Khan Academy's unit on Probability**.

Probability theory

- Some things in life *seem* random.
    - e.g., flipping a coin or rolling a die 🎲.
- The **probability** of seeing "heads" when flipping a fair coin is $\frac{1}{2}$.
- One interpretation of probability says that if we flipped a coin infinitely many times, then $\frac{1}{2}$ of the outcomes would be heads.

ex.: rolling die

Terminology

- **Experiment**: A process or action whose result is random.
    - e.g., rolling a die.
    - e.g., flipping a coin twice.
- **Outcome**: The result of an experiment.
    - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
    - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event**: A set of outcomes.
    - e.g., the event that the die lands on a even number is the set of outcomes {2, 4, 6}.
    - e.g., the event that the die lands on a 5 is the set of outcomes {5}.
    - e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.

1 2
3 4
5 6

roll even

Terminology

- **Probability**: A number between 0 and 1 (equivalently, between 0% and 100%) that describes the likelihood of an event.
    - 0: The event never happens.
    - 1: The event always happens.
- Notation: If A is an event, $P(A)$ is the probability of that event.

Equally-likely outcomes

- If all of the possible outcomes are equally likely, then the probability of A is

$$P(A) = \frac{\#\text{ of outcomes satisfying } A}{\text{total } \#\text{ of outcomes}}$$

- **Example 1**: Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

die

1 2
3 4
5 6

$P(\text{rolling even}) = \frac{3}{6}$ ← even #s ← possible #s

outcomes

HHH THH
HHT THT
HTH TTH
HTT TTT

$p(\text{exactly 2 H}) = \frac{3}{8}$

wrong:

outcomes

0H 1H
2H 3H

$P(2H) = \frac{1}{4}$

wrong bc outcomes not equally likely

Concept Check ✅ – Answer at **cc.dsc10.com**

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then – **without putting it back** – I choose another card at random and it is red?

without replacement

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

R B G

another way:

$P(\text{get } G) = \frac{1}{3}$

$P(\text{get } R \text{ on } 2^{nd} \text{ if got } G \text{ on } 1^{st}) = 1/2 \implies$

multiply $\frac{1}{3} * \frac{1}{2} = \frac{1}{6}$

outcomes

RB RG

BG BR

GB GR

$P(GR) = \frac{1}{6}$

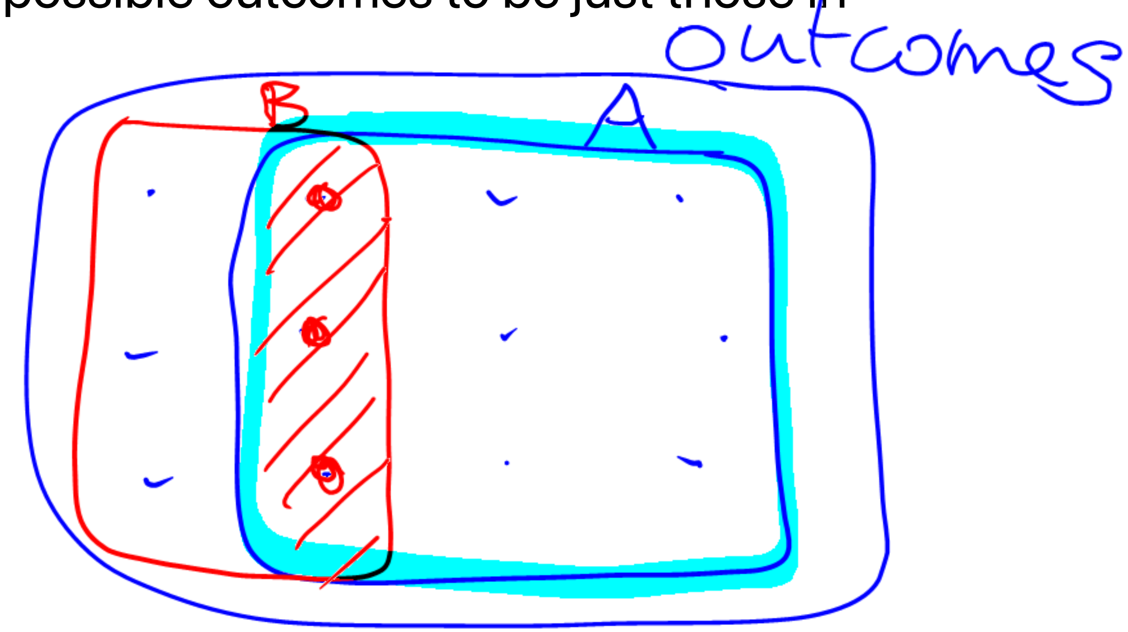
Conditional probabilities

- Two events A and B can both happen. Suppose that we know A has happened, but we don't know if B has.
- If all outcomes are equally likely, then the conditional probability of B given A is:

$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A} = \frac{3}{9} = \frac{1}{3}$$

if you know

- Intuitively, this is similar to the definition of the regular probability of B, $P(B) = \frac{\# \text{ of outcomes satisfying } B}{\text{total } \# \text{ of outcomes}}$, if you restrict the set of possible outcomes to be just those in event A.



told A happens ⇒ limit options to just A

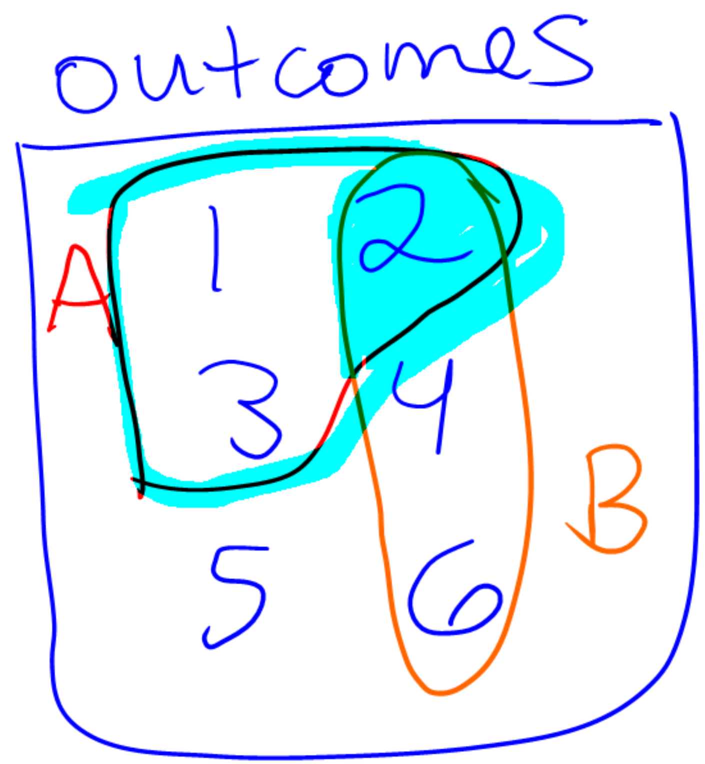
Concept Check ✅ – Answer at **cc.dsc10.com**

$$P(B \text{ given } A) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\#\text{ of outcomes satisfying } A} = \frac{1}{3}$$

I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

- A) $\frac{1}{2}$
- B) $\frac{1}{3}$
- C) $\frac{1}{4}$
- D) None of the above.

$$P(\underbrace{\text{even}}_{B} \text{ given } \underbrace{\leq 3}_{A})$$



outcomes

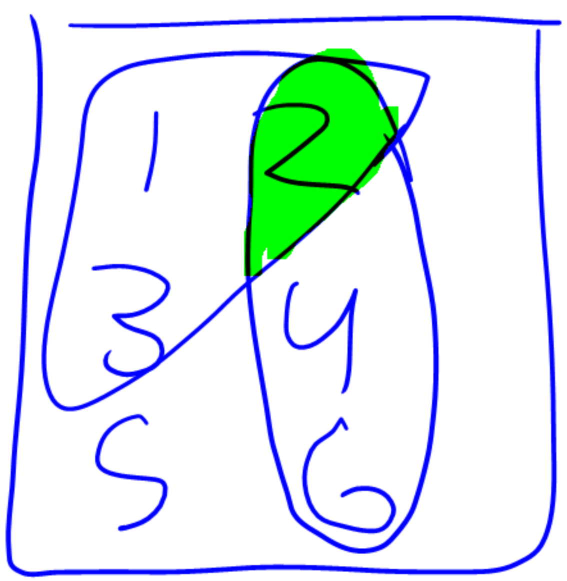
Probability that two events both happen

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that both A and B occur is

$$P(A \text{ and } B) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\text{total }\#\text{ of outcomes}}$$

- **Example 2**: I roll a fair six-sided die. What is the probability that the roll is 3 or less **and** even?

outcomes



$$\frac{1}{6}$$

The multiplication rule

- The multiplication rule specifies how to compute the probability of both A and B happening, even if all outcomes are not equally likely.

first A, then B given A

$$\underbrace{P(A \text{ and } B)} = P(A) \cdot \underbrace{P(B \text{ given } A)}$$

- **Example 2, again**: I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?

$$P(\underbrace{\le 3}_{A} \text{ and } \underbrace{\text{even}}_{B}) = \underbrace{P(\le 3)} \cdot \underbrace{P(\text{even given} \le 3)}_{\text{already calculated}}$$

$$\frac{3}{6} \cdot \frac{1}{3} \qquad \boxed{\begin{matrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{matrix}}$$

$$= 1/6$$

What if A isn't affected by B? 🤔

- The multiplication rule states that, for any two events A and B,

$$P(A \text{ and } B) = P(A) \cdot \boxed{P(B \text{ given } A)}$$

→ when A and B are independent, this is P(B)

- What if knowing that A happens doesn't tell you anything about the likelihood of B happening?
    - Suppose we flip a fair coin three times.
    - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?

ind: $P(A \text{ given } B) = P(A)$ → example: events: roll ≤ 2, roll even are independent

Independent events

- Two events A and B are independent if $P(B \text{ given } A) = P(B)$, or equivalently if

$$\boxed{P(A \text{ and } B) = P(A) \cdot P(B)}$$

- **Example 3**: Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

$P(\text{H on } 1^{st} \text{ AND H on } 2^{nd} \text{ AND} \ldots)$

$= P(\text{H on } 1^{st}) * P(\text{H on } 2^{nd}) * \ldots$

$= 0.7 * 0.7 * \ldots$

$= (0.7)^5 \leftarrow$ not $0.7 * 5$

dependent: roll ≤ 3, roll even

1	2
3	4
5	6

1	2
3	4
5	6

Probability that an event *doesn’t* happen

- The probability that A **doesn't** happen is $1 - P(A)$.
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.

how many outcomes? 8

N = no
Y = yes

outcomes
NNN ... YYN
YNY ... NNY

Concept Check ✅ – Answer at **cc.dsc10.com**

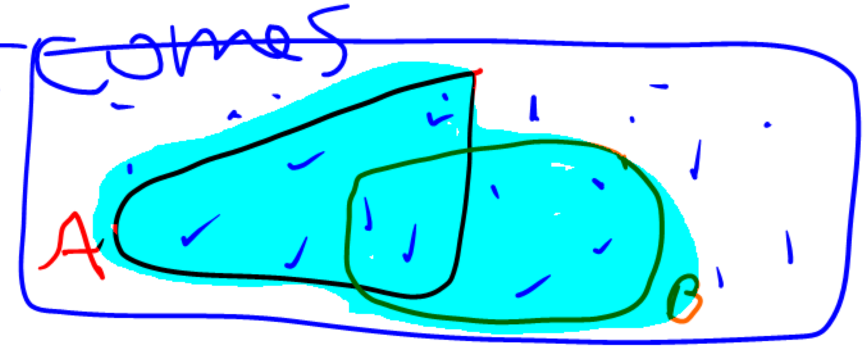
Every time I call my grandma 👵, the probability that she answers her phone is $\frac{1}{3}$, independently for each call. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- ~~A) $\frac{1}{3}$~~ should be $> \frac{1}{3}$ with multiple calls
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- ~~D) 1~~ no guarantee! < 1
- E) None of the above.

$P(\text{not answer}) = \frac{2}{3}$

$P(\text{not answer AND not answer AND not answer}) = \left(\frac{2}{3}\right)^3$

$\text{answer} = 1 - \left(\frac{2}{3}\right)^3 = 1 - \frac{8}{27} = \frac{27-8}{27} = \frac{19}{27}$

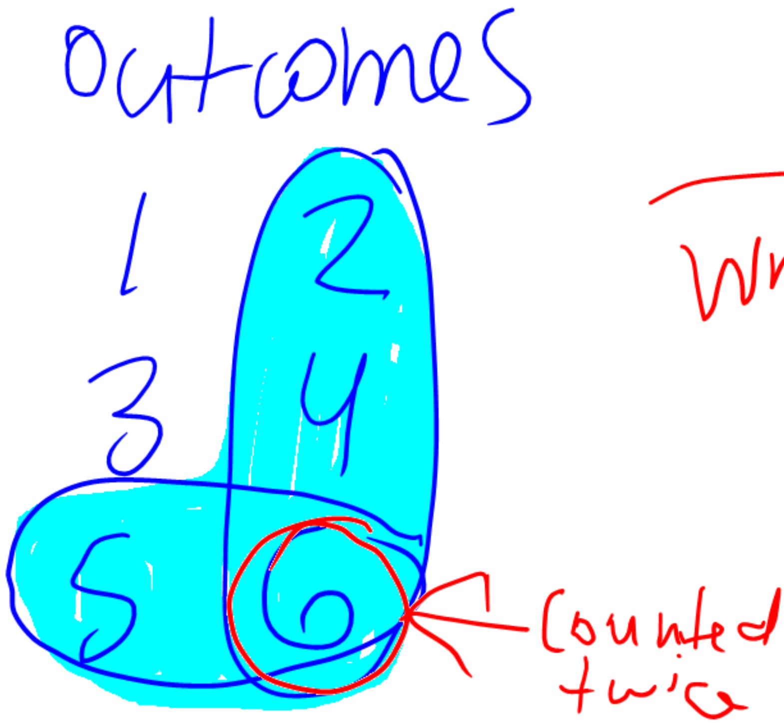


Probability of either of two events happening

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that either A or B occur is

$$P(A \text{ or } B) = \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \#\text{ of outcomes}}$$

- **Example 4**: I roll a fair six-sided die. What is the probability that the roll is even or at least 5?



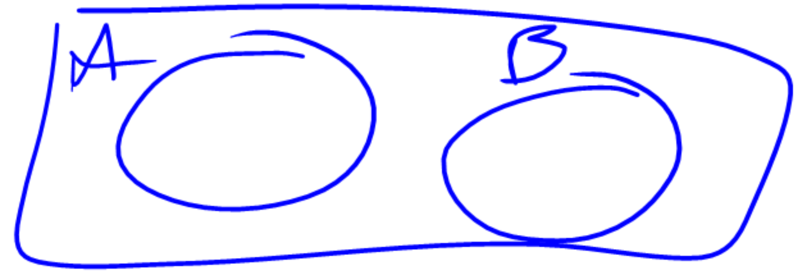
$$P = \frac{4}{6} = \frac{2}{3}$$

wrong: $P(A \text{ or } B) = P(A) + P(B)$

$= P(\text{even}) + P(\geq 5)$

$= \frac{3}{6} + \frac{2}{6} = 5/6$

not true in general



The addition rule

- Suppose that if A happens, then B doesn't, and if B happens, then A doesn't.
  - Such events are called **mutually exclusive** – they have **no overlap**.
- If A and B are any two mutually exclusive events, then

$$\boxed{P(A \text{ or } B) = P(A) + P(B)}$$

- **Example 5**: Suppose I have two biased coins, coin A and coin B. Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

$$P(\text{two diff}) = P(A\ H \text{ and } B\ T) \underbrace{\text{ or }}_{+} P(A\ T \text{ and } B\ H)$$

$$= 0.6 * 0.7 + 0.4 * 0.3$$

Aside: Proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$
\begin{aligned}
P(A \text{ or } B) &= \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total }\#\text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A) + (\#\text{ of outcomes satisfying } B)}{\text{total }\#\text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A)}{\text{total }\#\text{ of outcomes}} + \frac{(\#\text{ of outcomes satisfying } B)}{\text{total }\#\text{ of outcomes}} \\
&= P(A) + P(B)
\end{aligned}
$$

Summary, next time

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
    - The **multiplication rule**, which states that for any two events,

      $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$.
    - The **addition rule**, which states that for any two **mutually exclusive** events,

      $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** Simulations.