Lecture 11 – Probability

DSC 10, Fall 2024

Announcements

- Discussion is **today**. New policies starting in two weeks:
    - You must take and upload a photo with your ID for credit. No sign-in sheet option.
    - Submissions will close at **11:59PM** the day of discussion and we won't accept your photo outside of this.
- Quiz 2 is on **Wednesday** in your assigned quiz session.
    - Same time as Quiz 1, but new seats. You should get an email tomorrow with your seating assignment.
    - The quiz covers Lecture 5 through 10 and related labs and homeworks.
- Lab 3 is due **Thursday**. Homework 3 is due on **Sunday**.
    - Do as much of these assignments as possible before the quiz.

Agenda

We'll cover the basics of probability theory. This is a math lesson; take written notes ✍🏽.

Probability resources

Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5**.
- **Theory Meets Data, Chapters 1 and 2**.
- **Khan Academy's unit on Probability**.

Probability theory

- Some things in life *seem* random.
  - e.g., flipping a coin or rolling a die 🎲.
- The **probability** of seeing "heads" when flipping a fair coin is $\frac{1}{2}$.
- One interpretation of probability says that if we flipped a coin infinitely many times, then $\frac{1}{2}$ of the outcomes would be heads.

Terminology

- **Experiment**: A process or action whose result is random.
    - e.g., rolling a die.
    - e.g., flipping a coin twice.
- **Outcome**: The result of an experiment.
    - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
    - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event**: A set of outcomes.
    - e.g., the event that the die lands on a even number is the set of outcomes {2, 4, 6}.
    - e.g., the event that the die lands on a 5 is the set of outcomes {5}.
    - e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.

Terminology

- **Probability**: A number between 0 and 1 (equivalently, between 0% and 100%) that describes the likelihood of an event.
    - 0: The event never happens.
    - 1: The event always happens.
- Notation: If A is an event, $P(A)$ is the probability of that event.

Equally-likely outcomes

- If all of the possible outcomes are equally likely, then the probability of A is

$$P(A) = \frac{\#\text{ of outcomes satisfying } A}{\text{total }\#\text{ of outcomes}} = \frac{\#\text{ good}}{\text{total }\#}$$

- **Example 1**: Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

outcomes

HHH THH
HTH TTH
HHT THT
HTT TTT

$$\frac{3}{8}$$

incorrect

outcomes

0 H
1 H
2 H
3 H

~~1/4~~

not equally likely

$P(G \text{ on } 1^{st} \text{ and } R \text{ on } 2^{nd}) = P(G \text{ on } 1^{st}) \times P(R \text{ on } 2^{nd} \text{ given } G \text{ on } 1^{st})$

Concept Check ✅ – Answer at **cc.dsc10.com**

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then – **without putting it back** – I choose another card at random and it is red?

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

R B $\boxed{G}$ | R $\boxed{B}$

$\frac{1}{3} \times \frac{1}{2}$

elementary solution

RG GB BG

RB GR BR

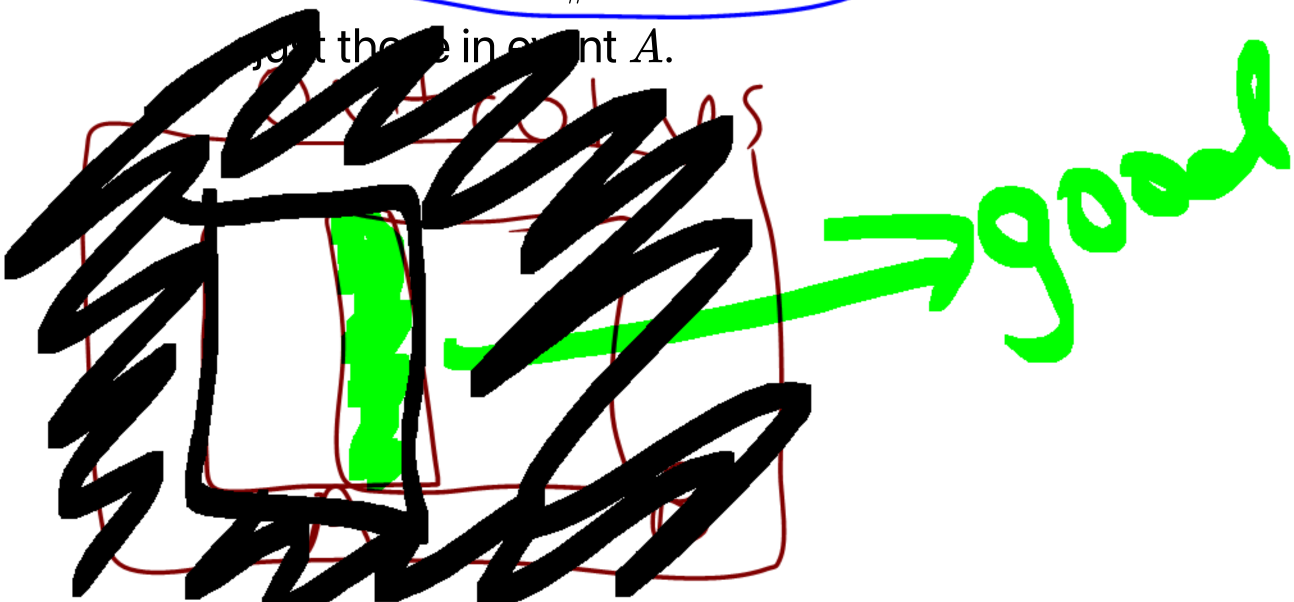
$\boxed{\frac{1}{6}}$

Conditional probabilities

- Two events A and B can both happen. Suppose that we know A has happened, but we don't know if B has.
- If all outcomes are equally likely, then the conditional probability of B given A is:

$$P(B \text{ given } A) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\#\text{ of outcomes satisfying } A} = \frac{\#\text{ good}}{\#\text{ total}}$$

- Intuitively, this is similar to the definition of the regular probability of B, $P(B) = \frac{\#\text{ of outcomes satisfying } B}{\text{total }\#\text{ of outcomes}}$, if you restrict the set of possible outcomes to be just those in event A.



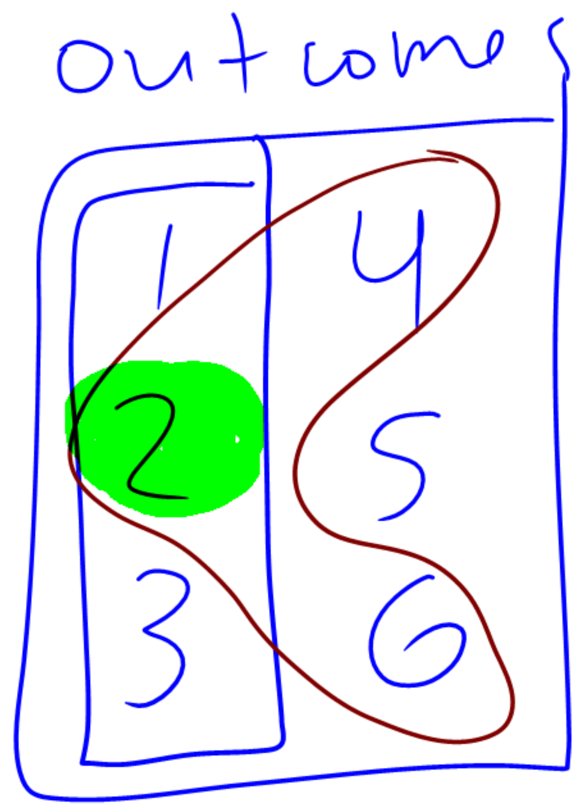
Concept Check ✅ – Answer at **cc.dsc10.com**

$$P(B \text{ given } A) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\#\text{ of outcomes satisfying } A} = \frac{1}{3}$$

I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

- A) $\frac{1}{2}$
- B) $\frac{1}{3}$
- C) $\frac{1}{4}$
- D) None of the above.

$$\frac{1}{3}$$

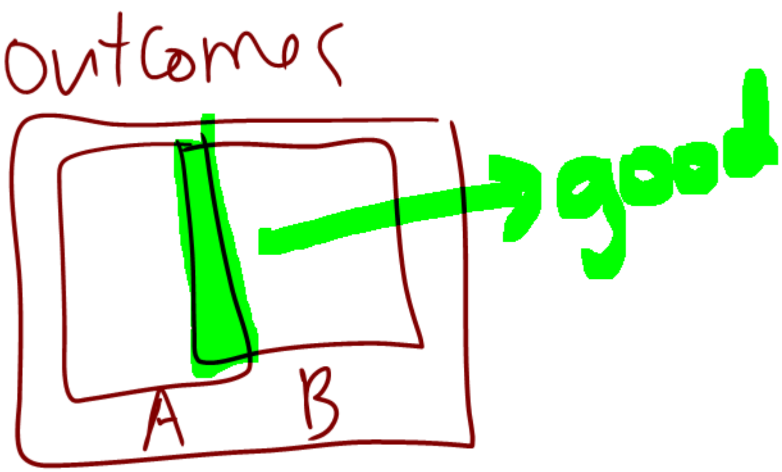


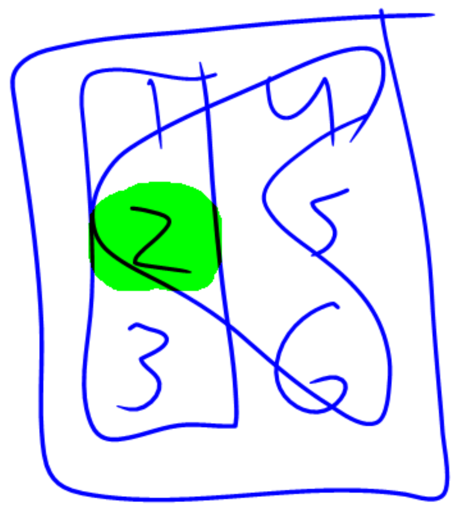
Probability that two events both happen

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that both A and B occur is

$$P(A \text{ and } B) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\text{total } \#\text{ of outcomes}}$$

- **Example 2**: I roll a fair six-sided die. What is the probability that the roll is 3 or less **and** even?





$$\frac{1}{6}$$

want both A and B

first, make A happen

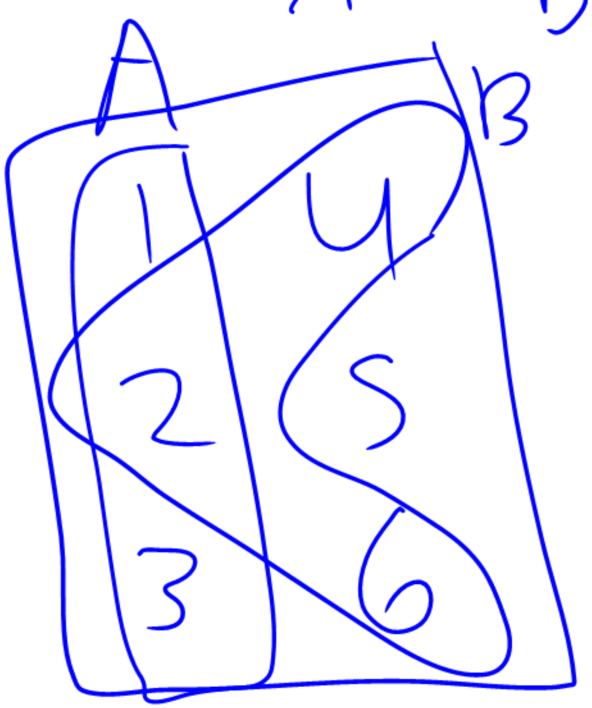
The multiplication rule

- The multiplication rule specifies how to compute the probability of both A and B happening, even if all outcomes are not equally likely.

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

then make B happen in the case that A happens

- **Example 2, again**: I roll a fair six-sided die. What is the probability that the roll is $\underbrace{\text{3 or less}}_{A}$ and $\underbrace{\text{even}}_{B}$?



$$P(A \text{ and } B)$$

$$= P(A) \times P(B \text{ given } A)$$

$$\frac{\cancel{3}}{6} \times \frac{1}{\cancel{3}} = \frac{1}{6}$$

What if A isn't affected by B? 🤔

- The multiplication rule states that, for any two events A and B,

$$P(A \text{ and } B) = P(A) \cdot \underbrace{P(B \text{ given } A)}$$

this simplifies to just $P(B)$ when A is not relevant to B (independent)

- What if knowing that A happens doesn't tell you anything about the likelihood of B happening?
  - Suppose we flip a fair coin three times.
  - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?

Independent events

- Two events A and B are independent if $P(B \text{ given } A) = P(B)$, or equivalently if

special case — not all the time, only when A and B are independent

true

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

and = × when events are independent

- **Example 3**: Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

$P(1^{st}\ H \text{ and } 2^{nd}\ H \text{ and} \ldots 5^{th}\ H)$

$0.7 \times 0.7 \times \ldots \times 0.7$

$(0.7)^5$

Probability that an event *doesn't* happen

- The probability that A **doesn't** happen is $1 - P(A)$.
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.

$$P\left(\begin{matrix}\text{talk}\\\text{at least}\\\text{once}\end{matrix}\right) = 1 - P\left(\begin{matrix}\text{never}\\\text{talk}\end{matrix}\right) = 1 - \frac{8}{27} = \boxed{\frac{19}{27}}$$

Concept Check ✅ – Answer at **cc.dsc10.com**

Every time I call my grandma 👵, the probability that she answers her phone is $\frac{1}{3}$, independently for each call. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- ~~A) $\frac{1}{3}$~~
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- ~~D) 1~~
- E) None of the above.

$\frac{1}{27} = \left(\frac{1}{3}\right)^3$

YYY

YNY

YYN

YNN

NYY

NNY

NYN

NNN bad

$\frac{8}{27} = \left(\frac{2}{3}\right)^3$

outcomes

YNY

how many outcomes? 8

all equally likely? NO

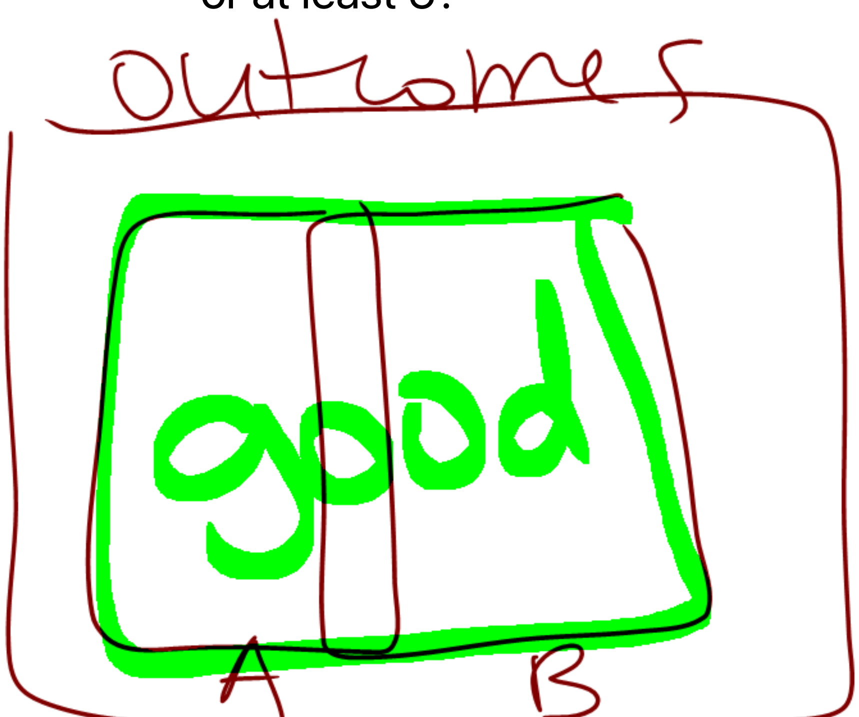
Not correct: $P(\text{even}) + P(\geq 5) = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$

Probability of either of two events happening

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that either A or B occur is

$$P(A \text{ or } B) = \frac{\# \text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}}$$

- **Example 4**: I roll a fair six-sided die. What is the probability that the roll is even or at least 5?



$\frac{4}{6} = \frac{2}{3}$

The addition rule

- Suppose that if A happens, then B doesn't, and if B happens, then A doesn't.
    - Such events are called **mutually exclusive** – they have **no overlap**.
- If A and B are any two mutually exclusive events, then

special case - not true always

$$P(A \text{ or } B) = P(A) + P(B)$$

or = + when mutually exclusive

- **Example 5**: Suppose I have two biased coins, coin A and coin B. Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

application: cases that don't overlap

case 1	case 2
$A \Rightarrow H$ and $B \Rightarrow T$	$A \Rightarrow T$, $B \Rightarrow H$

$$P(\text{2 diff faces}) = P(\text{case 1 or case 2})$$
$$= P(\text{case 1}) + P(\text{case 2})$$
$$= 0.6 \times 0.7 + 0.4 \times 0.3$$
$$= \boxed{0.54}$$

Aside: Proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$
\begin{aligned}
P(A \text{ or } B) &= \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A) + (\#\text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A)}{\text{total } \# \text{ of outcomes}} + \frac{(\#\text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\
&= P(A) + P(B)
\end{aligned}
$$

Summary, next time

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
    - The **multiplication rule**, which states that for any two events, $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$.
    - The **addition rule**, which states that for any two **mutually exclusive** events, $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** Simulations.